Mapping Fermion and Boson systems onto the Fock space of harmonic oscillators

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Abstract

The fluctuation-dissipation theorem (FDT) is very general and applies to a broad variety of different physical phenomena in condensed matter physics. With the help of the FDT and following the famous work of Caldeira and Leggett, we show that, whenever linear response theory applies, any generic bosonic or fermionic system at finite temperature T can be mapped onto a fictitious system of free harmonic oscillators. To the best of our knowledge, this is the first time that such a mapping is explicitly worked out. This finding provides further theoretical support to the phenomenological harmonic oscillator models commonly used in condensed matter. Moreover, our result helps in clarifying an interpretation issue related to the presence and physical origin of the Bose-Einstein factor in the FDT.

1 Introduction

The idea of modeling physical systems as a collection of harmonic oscillators has a long history and dates back to even before the birth of quantum mechanics. One of the best known example is Planck's work on black body radiation at the edge of the classical (beginning of quantum) era. More recently, this model has been of great importance in connection with the study of dissipation in quantum mechanics [1, 2, 3].

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In their famous paper devoted to the study of tunneling in dissipative systems [4], Caldeira and Legget observed that "any physical system which is weakly perturbed around its equilibrium state can be adequately represented (at T=0 at least) by regarding that system as equivalent to a set of simple harmonic oscillators". They supplemented this statement with an explicit computation, where they showed that given a quantum system at $T = 0$, in the lowest order approximation, its dynamics can be reproduced with the help of a properly constructed system of harmonic oscillators. They also stressed that the study of the $T \neq 0$ case (not considered in their paper) needed separate discussion.

Caldeira and Legget then suggested the form of the total lagrangian of a physical system in interaction with a certain "environment" (the "heat bath") and worked out the consequences of this assumption in connection with the tunneling problem [4, 5]. Prompted by this pioneering work, harmonic oscillator models are nowadays extensively used and there is little doubt that, from a phenomenological point of view, they reproduce quite well the physics of the systems under investigation. As a specific example, we can consider an electrical circuit, where the resistance is modeled with a collection of harmonically oscillating electrical dipoles.

From a theoretical point of view, however, it would be more satisfactory if we could prove that, for any generic system at finite temperature $T \neq 0$, it is possible to find an equivalent system of harmonic oscillators such that the statistical (thermodynamical) properties of the real physical system are properly reproduced by the system of oscillators. This would be an extension of the Caldeira Legget result and would provide further theoretical support to the commonly used phenomenological models.

The main scope of this work is to present a new and very general result which provides the above mentioned extension of the Caldeira-Legget one. By working within the framework of the FDT [6], we show that, whenever linear response theory applies, any generic bosonic and/or fermionic system at finite temperature can be mapped onto the Fock space of a fictitious system of free harmonic oscillators at the same temperature.

As a byproduct of our analysis, we shall see that our finding should help in clarifying an interpretation issue concerning the Bose-Einstein (BE) distribution factor which appears in the FDT. Actually, an often raised question concerns the physical meaning and/or origin of the BE factor which appears in the relation between the power spectrum of the fluctuating quantity and the corresponding generalized susceptibility. Sometimes this term is interpreted as due to an harmonic oscillator composition of the physical system under investigation. Such an interpretation, however, is not supported by the derivation of the theorem itself (see for instance [7, 8, 9]). Moreover, the FDT applies to any generic bosonic or fermionic system (irrespectively of its statistics).

Far from being an academic question, the resolution of this interpretation issue is of very practical importance in many different contexts [10, 11, 12, 13]. From a real understanding of the origin of this term often depends the correct physical interpretation of theoretical and experimental results [8, 9, 14, 15, 16, 17]. As we shall see, our results suggest that this term does not originate from underlying physical oscillator degrees of freedom of the system but is rather a general property related to the approximation (linear response) involved in the derivation of the FDT.

According to this theorem, whenever linear response theory is applicable, given a generic system which interacts with an external field $f(t)$ through the interaction term $\hat{V} = -f(t)\,\hat{A}$, where $\hat{A}$ is an observable of the system, the mean square of the Fourier transform $\hat{A}(\omega)$ of $\hat{A}(t)$ is related to the imaginary part $\chi''_A(\omega)$ of the corresponding (Fourier transformed) generalized susceptibility by the relation [6] :

$$\langle \hat{A}^2(\omega)\rangle = \hbar\chi''_A(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right) = 2\,\hbar\,\chi''_A(\omega)\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right), \tag{1}$$

where $\beta = 1/kT$, with T the temperature of the system and k the Boltzmann constant.

For instance, in the case of a resistively shunted Josephson junction [10], when applied to the power spectrum $S_I(\omega)$ of the noise current (fluctuation) in the resistive shunt (dissipation), the theorem takes the form (R is the shunt resistance)[11]:

$$S_I(\omega) = \frac{4}{R}\left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right). \tag{2}$$

The power spectrum $S_I(\omega)$ has been measured [10] and good agreement between the experimental results and Eq. (2) has been found.

The above $\frac{\hbar\omega}{2}$ term is sometimes presented [10, 11, 18, 19] as due to zero point energies and the experimental results [10] as a measurement of them. In fact, the term in parenthesis in Eq. (2) coincides with the mean energy of an harmonic oscillator of frequency ω in a thermal bath. The same holds true for the general case of Eq. (1), where the similar term is the mean energy of an harmonic oscillator in $\hbar\omega$ units, i.e. the BE distribution function. In the following we shall see that our result (the mapping) strongly suggests that the agreement between the experimental results [10] and Eq. (2) cannot be considered as a signature of measurement of zero point energies.

The rest of the paper is organized as follows. In section 2 we briefly review the derivation of the FDT and establish some relations useful for the following. In section 3 we establish our new result, the *mapping*, i.e. we show that for any given bosonic or fermionic system at finite temperature $T \neq 0$ we can always find a fictitious system of harmonic oscillators in such a manner that the physical quantities which appear in the FDT can be obtained from this equivalent system of oscillators. Section 4 is for our comments and conclusions. In particular, in this last section we present our comments on the interpretation issue related to the BE term in the FDT.

2   The fluctuation-dissipation theorem

Let us begin by briefly reviewing the derivation of the FDT. Consider a macroscopic system with unperturbed Hamiltonian $\hat{H}_0$ under the influence of the perturbation

$$\hat{V} = -f(t)\,\hat{A}(t)\,, \tag{3}$$

where $\hat{A}(t)$ is an observable (a bosonic operator) of the system and $f(t)$ an external generalized force[4]. Let $|E_n\rangle$ be the $\hat{H}_0$ eigenstates (with eigenvalues E_n) and $\langle E_n|\hat{A}(t)|E_n\rangle = 0$. Within the framework of linear response theory, the quantum-statistical average $\langle\hat{A}(t)\rangle_f$ of the observable $\hat{A}(t)$ in the presence of $\hat{V}$ is given by

$$\langle\hat{A}(t)\rangle_f = \int_{-\infty}^{t} dt' \chi_A(t-t') f(t') \tag{4}$$

where $\chi_A(t-t')$ is the generalized susceptibility,

$$\chi_A(t-t') = \frac{i}{\hbar}\theta(t-t')\langle[\hat{A}(t),\hat{A}(t')]\rangle = -\frac{1}{\hbar}G_R(t-t')\,, \tag{5}$$

with $\langle ... \rangle = \sum_n \varrho_n \langle E_n|...|E_n\rangle$, $\varrho_n = e^{-\beta E_n}/Z$, $Z = \sum_n e^{-\beta E_n}$, $G_R(t-t')$ being the retarded Green's function and $\hat{A}(t) = e^{i\hat{H}_0 t/\hbar}\hat{A}e^{-i\hat{H}_0 t/\hbar}$.

Defining the correlators (from now on $t' = 0$):

$$G_>(t) = \langle\hat{A}(t)\,\hat{A}(0)\rangle \qquad \text{and} \qquad G_<(t) = \langle\hat{A}(0)\,\hat{A}(t)\rangle\,, \tag{6}$$

so that $G_R(t) = -i\theta(t)(G_>(t) - G_<(t))$, and the corresponding Fourier transforms, $G_>(\omega)$ and $G_<(\omega)$ respectively, it is a matter of few lines to show that:

$$G_>(\omega) = -\frac{2}{1-e^{-\beta\hbar\omega}}\mathrm{Im}\,G_R(\omega) \qquad ; \qquad G_<(\omega) = e^{-\beta\hbar\omega}\,G_>(\omega)\,. \tag{7}$$

Finally, by noting that

$$\langle\hat{A}^2(\omega)\rangle = \frac{1}{2}(G_>(\omega) + G_<(\omega)) \tag{8}$$

and that the Fourier transform of $\chi_A(t)$ is $\chi_A(\omega) = \chi'_A(\omega) + i\chi''_A(\omega) = -\frac{1}{\hbar}G_R(\omega)$ we get:

$$\langle\hat{A}^2(\omega)\rangle = \hbar\chi''_A(\omega)\frac{1+e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \hbar\chi''_A(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right) = 2\,\hbar\chi''_A(\omega)\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right) \tag{9}$$

which is Eq. (1), the celebrated FDT.

As observed by Kubo et al. [7] (and shown in the derivation sketched above), the BE factor is simply due to a peculiar combination of Boltzmann factors in Eq. (9) and there is no reference to physical harmonic oscillators of the system whatsoever. Despite such an authoritative remark, some people insist in interpreting the $\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right)$ term as related to harmonic oscillator degrees of freedom of the physical system.

In the case of the measured [10] power spectrum of Eq. (2), some authors [12, 13] interpret this term as due to the electromagnetic field in the resistive shunt and therefore

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[4]More generally, we could consider a local observable and a local generalized force, in which case we would have $\hat{V} = -\int d^3\,\vec{r}\hat{A}(\vec{r})f(\vec{r},t)$, and successively define a local susceptibility $\chi(\vec{r},t;\vec{r'},t')$ (see Eq. (5) below). As this would add nothing to our argument, we shall restrict ourselves to $\vec{r}$-independent quantities. The extension to include local operators is immediate.

the first term in parenthesis of Eq. (2) as originating from zero point energies of this electromagnetic field. Such an interpretation, however, is not supported by any physical derivation and has been strongly criticized in [8, 14, 15, 16]. Very recently, starting from the results of the present work, we have also carefully investigated this issue [9], providing arguments which strongly support previous criticisms (see [9] for details).

Let us go back now to our analysis. For our purposes, it is useful to show that from Eqs. (5) and (8) we can easily derive the following expressions for $\chi_A''(\omega)$ and $\langle \hat{A}^2(\omega) \rangle$:

$$\chi_A''(\omega) = \frac{\pi}{\hbar} \sum_{i,j} \varrho_i |A_{ij}|^2 \left[\delta \left(\frac{E_i - E_j}{\hbar} + \omega \right) - \delta \left(\frac{E_j - E_i}{\hbar} + \omega \right) \right] , \tag{10}$$

and

$$\langle \hat{A}^2(\omega) \rangle = \pi \sum_{i,j} \varrho_i |A_{ij}|^2 \left[\delta \left(\frac{E_i - E_j}{\hbar} + \omega \right) + \delta \left(\frac{E_j - E_i}{\hbar} + \omega \right) \right] , \tag{11}$$

where $A_{ij} = \langle E_i | \hat{A} | E_j \rangle$.

In fact, by inserting in Eq. (5) the expressions:
$\theta(t - t') = - \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega(t-t')}}{\omega + i\eta}$, $I = \sum_i |E_i\rangle\langle E_i|$ and $\hat{A}(t) = e^{i\hat{H}_0 t/\hbar} \hat{A} e^{-i\hat{H}_0 t/\hbar}$ we get:

$$\chi_A(t - t') = -\frac{1}{\hbar} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega + i\eta} \sum_{i,j} \varrho_i |A_{ij}|^2 \left(e^{i(E_i - E_j)(t-t')/\hbar} - e^{-i(E_i - E_j)(t-t')/\hbar} \right) \tag{12}$$

Then, making use of $\lim_{\eta \to 0} \frac{1}{\omega + i\eta} = \mathcal{P} \left(\frac{1}{\omega} \right) - i\pi\delta(\omega)$, Eq. (10) follows immediately.

As for Eq. (11), from Eq. (6) for $G_>(t)$, we have:

$$G_>(t) = \sum_{i,j} \rho_i \langle E_i | e^{\frac{i}{\hbar}\hat{H}t} \hat{A} e^{-\frac{i}{\hbar}\hat{H}t} | E_j \rangle \langle E_j | \hat{A} | E_i \rangle = = \sum_{i,j} \rho_i e^{-\frac{i}{\hbar}(E_j - E_i)t} |A_{ij}|^2 .$$

Working out the similar expression for $G_<(t)$, for the correlation function $G(t)$ we get:

$$G(t) = \frac{1}{2}(G_>(t) + G_<(t)) = \frac{1}{2} \sum_{i,j} \rho_i |A_{ij}|^2 (e^{-\frac{i}{\hbar}(E_j - E_i)t} + e^{-\frac{i}{\hbar}(E_i - E_j)t}) , \tag{13}$$

so that the Fourier transform $\tilde{G}(\omega)$ is:

$$\tilde{G}(\omega) = \pi \sum_{i,j} \rho_i |A_{ij}|^2 \left[\delta \left(\frac{E_j - E_i}{\hbar} + \omega \right) + \delta \left(\frac{E_i - E_j}{\hbar} + \omega \right) \right] . \tag{14}$$

As $\langle \hat{A}^2 \rangle = G(0)$, $\tilde{G}(\omega)$ is the spectral density $\langle \hat{A}^2(\omega) \rangle$ of $\langle \hat{A}^2 \rangle$ ($\langle \hat{A}^2 \rangle = \int_{-\infty}^{+\infty} \langle \hat{A}^2(\omega) \rangle \frac{d\omega}{2\pi}$). Then, making use of Eq. (14), we finally get Eq. (11).

For our purposes, it is also useful to write Eq. (11) in a different manner. After some straightforward manipulations, Eq. (11) can be written as:

$$\langle \hat{A}^2(\omega) \rangle = \pi \sum_{j>i} (\varrho_i - \varrho_j) |A_{ij}|^2 \coth \left(\frac{\beta\hbar\omega_{ji}}{2} \right) [\delta(\omega - \omega_{ji}) + \delta(\omega + \omega_{ji})] \tag{15}$$

$$= \pi \coth \left(\frac{\beta\hbar\omega}{2} \right) \sum_{j>i} (\varrho_i - \varrho_j) |A_{ij}|^2 [\delta(\omega - \omega_{ji}) - \delta(\omega + \omega_{ji})] , \tag{16}$$

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where we have introduced the notation: $\omega_{ji} = \frac{E_j - E_i}{\hbar}$. By following similar steps, Eq. (10) can also be written as:

$$\chi''(\omega) = \frac{\pi}{\hbar} \sum_{j>i} (\varrho_i - \varrho_j) |A_{ij}|^2 \left[\delta\left(\omega - \omega_{ji}\right) - \delta\left(\omega + \omega_{ji}\right)\right] . \tag{17}$$

Clearly, comparing Eq. (16) with Eq. (17), we find, as we should, the FDT theorem.

Now, starting from Eqs. (15) and (17) and taking inspiration from the seminal work of Caldeira and Leggett [4], we shall be able to establish a formal mapping between the real system considered so far and a system of fictitious harmonic oscillators. A similar mapping, restricted however to the $T = 0$ case, was considered in [4], where it was also noted that the $T \neq 0$ case needs separate discussion. The mapping that we are going to construct in the present work deals with the $T \neq 0$ general case.

3 Mapping Boson and Fermion systems onto harmonic oscillators

To prepare the basis for the construction of this mapping, let us consider first a real system $\mathcal{S}_{osc}$ of harmonic oscillators (each of which is labeled below by the double index $\{ji\}$ for reasons that will become clear in the following) whose free Hamiltonian is:

$$\hat{H}_{osc} = \sum_{j>i} \left(\frac{\hat{p}_{ji}^2}{2M_{ji}} + \frac{M_{ji}\omega_{ji}^2}{2} \hat{q}_{ji}^2 \right) , \tag{18}$$

where ω_{ji} are the proper frequencies of the individual harmonic oscillators and M_{ji} their masses. Let $|n_{ji}\rangle$ $(n_{ji} = 0, 1, 2, ...)$ be the occupation number states of the $\{ji\}$ oscillator out of which the Fock space of $\mathcal{S}_{osc}$ is built up. Let us consider also $\mathcal{S}_{osc}$ in interaction with an external system through the one-particle operator:

$$\hat{V}_{osc} = -f(t)\hat{A}_{osc} , \tag{19}$$

with

$$\hat{A}_{osc} = \sum_{j>i} \left(\alpha_{ji} \hat{q}_{ji}\right) . \tag{20}$$

Obviously, the FDT applied to $\mathcal{S}_{osc}$ gives $\langle \hat{A}_{osc}^2(\omega)\rangle = \hbar \chi''_{osc}(\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right)$, but this is not what matters to us.

What is important for our purposes is that, differently from any other generic system, for $\mathcal{S}_{osc}$ we can exactly compute $\langle \hat{A}_{osc}^2(\omega)\rangle$ and $\chi''_{osc}(\omega)$ from Eqs. (10) and (11) because we can explicitly compute the matrix elements of $\hat{A}_{osc}$.

In fact, if we apply Eqs. (13) and (14) to $\mathcal{S}_{osc}$ and replace the double index notation $(\omega_{ji} ; n_{ji} ; M_{ji} ;$ etc.) with the more convenient (for the time being) and self explanatory

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one index notation (ω_1, ω_2, ... ; n_1, n_2, ... ; M_1, M_2, ... ; etc.), for $\langle \hat{A}^2_{osc}(\omega) \rangle$ we have:

$$\langle \hat{A}^2_{osc}(\omega) \rangle = \pi \sum_{n_1,n_2,..} \sum_{m_1,m_2,..} (\varrho_{n_1} \varrho_{n_2} \cdots) |\langle n_1, n_2, .. | \hat{A}_{osc} | m_1, m_2, .. \rangle|^2$$
$$\times [\delta (\omega + l_1\omega_1 + l_2\omega_2 + \cdots) + \delta (\omega - l_1\omega_1 - l_2\omega_2 - \cdots)] , \quad (21)$$

where $l_k = n_k - m_k$, $\varrho_{n_k} = e^{-\beta(n_k+1/2)\hbar\omega_k}/Z_k$, $Z_k = \sum_{n_k} e^{-\beta(n_k+1/2)\hbar\omega_k}$ (note also that in this one index notation $\hat{A}_{osc}$ is written as $\hat{A}_{osc} = \sum_k (\alpha_k \hat{q}_k)$). Now, as

$$\langle n_k | \hat{q}_k | m_k \rangle = \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1} \langle n_k + 1 | m_k \rangle + \sqrt{n_k} \langle n_k - 1 | m_k \rangle \right) , \quad (22)$$

we immediately get:

$$\langle \hat{A}^2_{osc}(\omega) \rangle = \pi \sum_{n_1,n_2,..} \sum_{m_1,m_2,..} (\varrho_{n_1} \varrho_{n_2} \cdots)$$
$$\times \left[\sum_k \alpha_k \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1}\delta_{m_k,n_k+1} + \sqrt{n_k}\delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} \right]^2$$
$$\times [\delta (\omega + l_1\omega_1 + l_2\omega_2 + \cdots) + \delta (\omega - l_1\omega_1 - l_2\omega_2 - \cdots)] . \quad (23)$$

Let us concentrate our attention to the square in the second line of Eq. (23). Due to the presence of the Kronecker deltas, all the crossed terms in this square, i.e. all the terms with different values of the index k, vanish. In other words, the square of the sum is equal to the sum of the squares:

$$\left[\sum_k \alpha_k \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1}\delta_{m_k,n_k+1} + \sqrt{n_k}\delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} \right]^2$$
$$= \sum_k \left(\alpha_k \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1}\delta_{m_k,n_k+1} + \sqrt{n_k}\delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} \right)^2 \quad (24)$$

For the same reason, the same holds true for each value of the index k, i.e.:

$$\left(\alpha_k \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1}\delta_{m_k,n_k+1} + \sqrt{n_k}\delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} \right)^2$$
$$= \alpha_k^2 \frac{\hbar}{2M_k\omega_k} \left((n_k + 1) \delta_{m_k,n_k+1} + n_k \delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} . \quad (25)$$

Therefore, as $l_k = n_k - m_k$, for $\langle \hat{A}^2_{osc}(\omega) \rangle$ we get:

$$\langle \hat{A}^2_{osc}(\omega) \rangle = \pi \sum_{n_1,n_2,..} (\varrho_{n_1} \varrho_{n_2} \cdots) \sum_k \alpha_k^2 \frac{\hbar}{2M_k\omega_k} (2 n_k + 1) (\delta(\omega - \omega_k) + \delta(\omega + \omega_k)) . \quad (26)$$

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Finally, as $\sum_{n_k} \varrho_{n_k} = 1$, the above expression becomes:

$$\langle \hat{A}_{osc}^2(\omega) \rangle = \pi \sum_k \alpha_k^2 \frac{\hbar}{2M_k\omega_k} (\delta(\omega - \omega_k) + \delta(\omega + \omega_k)) \sum_{n_k} \varrho_{n_k} (2 n_k + 1) \qquad (27)$$

$$= \pi \sum_k \alpha_k^2 \frac{\hbar}{2M_k\omega_k} \coth\left(\frac{\beta\hbar\omega_k}{2}\right) (\delta(\omega - \omega_k) + \delta(\omega + \omega_k)). \qquad (28)$$

Going back to the original double index notation:

$$\langle \hat{A}_{osc}^2(\omega) \rangle = \pi \sum_{j>i} \alpha_{ji}^2 \frac{\hbar}{2M_{ji}\omega_{ji}} \coth\left(\frac{\beta\hbar\omega_{ji}}{2}\right) (\delta(\omega - \omega_{ji}) + \delta(\omega + \omega_{ji})) \qquad (29)$$

$$= \pi \coth\left(\frac{\beta\hbar\omega}{2}\right) \sum_{j>i} \alpha_{ji}^2 \frac{\hbar}{2M_{ji}\omega_{ji}} (\delta(\omega - \omega_{ji}) - \delta(\omega + \omega_{ji})). \qquad (30)$$

We have just seen that given a real system $\mathcal{S}_{osc}$ of harmonic oscillators and the one particle operator $\hat{A}_{osc}$ of Eq. (20), for such an operator is possible to evaluate explicitly $\langle \hat{A}_{osc}^2(\omega) \rangle$. We find that each of the individual harmonic oscillators gives rise to a term $\coth\left(\frac{\beta\hbar\omega_{ji}}{2}\right)$ which in turn comes from the term $\sum_{n_{ji}} \varrho_{n_{ji}} (2 n_{ji} + 1)$ of Eq. (27).

Let us now consider $\chi_{osc}''(\omega)$, which (see Eqs. (10) and (23)) is nothing but:

$$\chi_{osc}''(\omega) = \frac{\pi}{\hbar} \sum_{n_1,n_2,..} \sum_{m_1,m_2,..} (\varrho_{n_1} \varrho_{n_2} \cdots)$$
$$\times \left[\sum_k \alpha_k \sqrt{\frac{\hbar}{2M_k\omega_k}} \left(\sqrt{n_k + 1}\delta_{m_k,n_k+1} + \sqrt{n_k}\delta_{m_k,n_k-1} \right) \prod_{h \neq k} \delta_{m_h,n_h} \right]^2$$
$$\times \left[\delta(\omega + l_1\omega_1 + l_2\omega_2 + \cdots) - \delta(\omega - l_1\omega_1 - l_2\omega_2 - \cdots) \right]. \qquad (31)$$

Apart from the factor $1/\hbar$, Eq. (31) differs from Eq. (23) because it contains the difference (rather than the sum) of delta functions in the last line.

If we proceed for $\chi_{osc}''(\omega)$ as we have just done for $\langle \hat{A}_{osc}^2(\omega) \rangle$, we immediately note that the only difference with the previous computation is due to this minus sign. In fact, its presence causes that rather than the combination $(2 n_k + 1)$ of Eq. (26), which comes from the sum $(n_k + 1) + n_k$ of Eq. (25), we get the combination $(n_k + 1) - n_k = 1$. Therefore, for $\chi_{osc}''(\omega)$ we do not get the sum $\sum_{n_k} \varrho_{n_k} (2 n_k + 1) = \coth\left(\frac{\beta\hbar\omega_k}{2}\right)$ of Eq. (27), but rather $\sum_{n_k} \varrho_{n_k} = 1$. Then:

$$\chi_{osc}''(\omega) = \frac{\pi}{\hbar} \sum_{j>i} \alpha_{ji}^2 \frac{\hbar}{2M_{ji}\omega_{ji}} (\delta(\omega - \omega_{ji}) - \delta(\omega + \omega_{ji})). \qquad (32)$$

Naturally, comparing Eq. (30) with Eq. (32) we see that for $\mathcal{S}_{osc}$ the FDT holds true, as it should. However, what is important for our purposes is to note that for this system we have been able to compute separately $\langle \hat{A}_{osc}^2(\omega) \rangle$ and $\chi_{osc}''(\omega)$ and found that the $\coth\left(\frac{\beta\hbar\omega}{2}\right)$

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factor of the FDT originates from the individual contributions $\coth \left(\frac{\beta \hbar \omega_{ji}}{2} \right)$ of each of the harmonic oscillators of $\mathcal{S}_{osc}$.

We are now in the position to build up our mapping. Let us consider the original system $\mathcal{S}$, described by the unperturbed Hamiltonian $\hat{H}_0$, in interaction with an external field $f(t)$ through the interaction term $\hat{V} = -f(t)\,\hat{A}$ (see Eq. (3)), and construct a fictitious system of harmonic oscillators $\mathcal{S}_{osc}$, described by the free Hamiltonian $\hat{H}_{osc}$ of Eq. (18), in interaction with the same external field $f(t)$ through the interaction term $\hat{V}_{osc}$ of Eq. (19), with $\hat{A}_{osc}$ given by Eq. (20), where for α_{ji} we choose:

$$\alpha_{ji} = \left(\frac{2M_{ji}\omega_{ji}}{\hbar} \right)^{\frac{1}{2}} (\varrho_i - \varrho_j)^{\frac{1}{2}} |A_{ij}| \tag{33}$$

and for the proper frequencies ω_{ji} of the oscillators:

$$\omega_{ji} = (E_j - E_i)/\hbar > 0, \tag{34}$$

with E_i the eigenvalues of the Hamiltonian $\hat{H}_0$ of the real system.

Comparing Eq. (30) with Eq. (16) and Eq. (32) with Eq. (17), it is immediate to see that with the above choices of α_{ji} and ω_{ji} we have:

$$\langle \hat{A}^2(\omega) \rangle = \langle \hat{A}^2_{osc}(\omega) \rangle \qquad \text{and} \qquad \chi''_A(\omega) = \chi''_{osc}(\omega). \tag{35}$$

Eqs. (33) and (34) are the central results of our analysis. Actually, these are the equations which allow to establish our mapping. In fact, with such a choice of the α's and the ω's, we are able to map the real system $\mathcal{S}$ onto a fictitious system of harmonic oscillators $\mathcal{S}_{osc}$,

$$\mathcal{S} \;\to\; \mathcal{S}_{osc}, \tag{36}$$

in such a manner that $\chi''_A(\omega)$ and $\langle \hat{A}^2(\omega) \rangle$ of the real system are equivalently obtained by computing the corresponding quantities of the fictitious one (Eqs. (35)).

This is the desired result. What we have just shown is that any generic boson or fermion system at finite temperature T is equivalent to a system of harmonic oscillators at the same temperature. From a theoretical point of view, the relevance of such a result should be immediately clear. As we have already observed, in fact, harmonic oscillator models are quite common in modeling generic physical systems. Now, the typical physical situation we have to deal with is that of a system (bosonic or fermionic) at finite temperature $T \neq 0$. In this respect, our mapping fills up the gap mentioned by Caldeira and Legget (see Appendix C of [4]) by extending the $T = 0$ mapping put forward by them to the general finite temperature case, thus providing further theoretical support to the use of these models.

In the following section, we would like to add some more comments on the above results. In particular, we are going to consider the previously mentioned interpretation issue concerning the presence and origin of the BE distribution factor in the FDT.

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4 Comments and conclusions

First of all, we point out that, in order to construct the above mapping, the key ingredient we used is the hypothesis that linear response theory is applicable, which is the main hypothesis under which the FDT is established. When this is not the case, Eq.(4) cannot be derived and we do not arrive to Eqs.(10) and (11), which are crucial to build up our mapping.

Note now that, by considering the "equivalent" harmonic oscillators system $\mathcal{S}_{osc}$ rather than the real one, we are somehow allowed to regard the BE distribution factor $\coth\left(\frac{\beta\hbar\omega}{2}\right)$ of the FDT in Eq.(1) as originating from the individual contributions $\coth\left(\frac{\beta\hbar\omega_{ji}}{2}\right)$ of each of the oscillators of the fictitious system (see above, Eqs.(29), (30) and (32)). In this sense, such a mapping allows for an oscillator interpretation of the BE term in the FDT.

At the same time, however, our result shows that this BE factor does not describe the physics of the system, i.e. it does not encode any real, physical, harmonic oscillator degrees of freedom of the system (see also the considerations below).

In this respect, it is worth to point out that what we have implemented is not a canonical transformation, i.e. it is not a transformation which allows to describe the system in terms of new degrees of freedom (such as normal modes), but a formal mapping, a mathematical construct, which can be established, we repeat ourselves, only within the framework of linear response theory.

In our opinion, then, our finding provides an answer to the questions of the "physical meaning" or "physical origin" of the BE term in the FDT or, stated differently, to the question of whether this BE distribution factor possibly describes the physical nature of the system or not [17].

In fact, from the derivation of the FDT, we know that the BE factor derives from a peculiar combination of Boltzmann factors (see [7] and Eq.(9) above). At the same time, we have shown that, regardless the bosonic or fermionic nature of the (real) system $\mathcal{S}$, it is always possible to establish a mapping which relates $\mathcal{S}$ to a system of harmonic oscillators $\mathcal{S}_{osc}$ so that this BE factor can be regarded as "originating" from the individual oscillators of the "equivalent" system $\mathcal{S}_{osc}$. Therefore, it is not the physical nature of the system which is encoded in this BE term but rather a fundamental quantum property of any bosonic and/or fermionic system: *whenever linear response theory is applicable, any generic system is, at least with respect to the FDT, equivalent (in the sense defined above) to a system of quantum harmonic oscillators.*

Before ending this section, it is probably worth to spend few words on some examples of realistic systems where our mapping is at work. In this respect, we would like to note that there are several applications in the literature where fermionic systems, after bosonization, are actually described by a system in interaction with a bosonic bath. This is, for instance, the case of the anisotropic Kondo model, which is shown to be equivalent to a spin-boson model (a two level system in interaction with a bosonic bath). The same is also true for a quantum dot interacting with external leads.

In the above examples, the system is described with the help of a spin-boson Hamil-

tonian, thus providing concrete realizations of the mapping discussed in this work. What amounts to the same thing, they are worked out examples where the Caldeira-Legget model is explicitly derived.

In this respect, in fact, it is important to note that any application of our mapping is concretely substantiated in the Caldeira-Leggett model. At the same time, we stress again that the present work is focused on the question of deeply understanding what is really behind the fact that this modelization is so successful in covering the essential features of dissipative systems. We believe that we achieve this goal by performing a thorough analysis of the fluctuation-dissipation theorem.

In summary, we have found that when linear response theory applies, any generic system can be mapped onto a fictitious system of harmonic oscillators so that that the mean square $\langle \hat{A}^2(\omega) \rangle$ of the fluctuating observable and the corresponding imaginary part of the generalized susceptibility $\chi_A''(\omega)$ of the real system are given by the corresponding quantities of the fictitious one. Moreover, we have seen that such a mapping allows to consider the BE distribution factor which appears in the FDT as originating from the individual harmonic oscillators of the fictitious equivalent system. This strongly suggests that it is only in this sense that this BE factor can be interpreted in terms of harmonic oscillators and that no other physical meaning can be superimposed to it.

We believe that our mapping has a broader range of applicability than the worked case of the FDT discussed in this paper. Work is in progress in this direction.

Acknowledgements

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